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Adaptive Moving Mesh Methods

Springer

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To our families: Fang, Eileen, Andrew and Robin, Nathan, Claire, Maria, and Simon.

Preface

Solutions of partial differential equations (PDEs) arising in science and engineering frequently have large variations occurring over small portions of the physical domain, and a major challenge when solving such problems is to appropriately resolve the solution behavior there. When finite difference or finite element methods are employed, a fine mesh is required in those particular parts of the physical domain. Using a uniform mesh throughout the physical domain can become a substantial, indeed a formidable, computational expense, especially in multidimensions where the number of mesh points required can be prohibitively large. A practical and often indispensable alternative is to place a high proportion of mesh points in the regions of large solution variation and few points in the rest of the domain. With this basic idea of *mesh adaptivity*, the total number of mesh points required is much smaller than with a uniform mesh, and significant economies are achieved.

The purpose of this book is to present the theoretical and practical aspects of mesh adaptivity, with particular emphasis on its application to time-dependent PDEs. Given the ubiquitous need for mesh adaptivity in the various areas of science and engineering, a proliferation of methods have been developed in the past. While on one hand this makes the study of mesh adaptivity an exciting, multifaceted endeavor, it has also made it a daunting task for the potential user of adaptive mesh techniques to know where to begin looking for a method suitable for his or her particular needs. In this book a major effort is made to make the general topic of mesh adaptivity more accessible, both practically and theoretically, to a broad audience. The intent, however, is not to provide a review of all adaptive mesh methods, but rather a detailed discussion of one type of method, the r-adaptive mesh method or the moving mesh method, which reflects the authors' research interest in this field over the past 20 years. Fortuitously, the fundamental principles presented are applicable in a much wider range of contexts, and there are often close (often complementary) relationships between the moving mesh method considered here and other important types of adaptive mesh methods such as the h- and p-adaptive methods.

It is important to say at the outset that mesh adaptivity has its place and should not be viewed as a panacea. For problems with smooth solutions, a uniform mesh suffices and is usually preferred over a nonuniform one because it readily lends itself to efficient solution. Even when adaptivity is required, an isotropic mesh can be preferred for the same reason if it can resolve the solution without using an undue number of mesh points. Anisotropic meshes are favored when there is a need for better alignment of the mesh with certain solution directions, such as those arising due to boundary or interior layers and sharp interfaces. The distinction between isotropic and anisotropic meshes is an important theme in the chapters on multidimensional adaptation.

The type of adaptive mesh methods considered in this book for solving PDEs is characterized by the fact that a mesh moves continuously in time while adapting to the evolving structures in the solution. These are called adaptive moving mesh methods, or simply *moving mesh methods* for short. Since point locations are dynamically *relocated* during the course of numerical computation, a moving mesh method is also called an r-adaptive method in the finite element community. The analysis of adaptivity within the moving mesh context focuses on how to optimally choose mesh points, where the computational cost is normally correlated with the number of mesh points used. The main goals of this book are to carry out this adaptivity analysis, understand the existing methods, and develop new ones, by relying on two key tools, the so-called *equidistribution* and *alignment* conditions. While the concept of equidistribution has been well-known and used in the mesh adaptivity community for many years, an understanding of alignment in multidimensions is relatively recent. These tools are discussed in detail in Chapters 2 and 4.

It is useful to note that the formulations of the moving mesh strategies presented in this book are generally independent of specific types of physical PDEs being solved (although discretization of the physical PDEs themselves are obviously not). Throughout the book, moving mesh methods are described mainly for parabolic PDEs. When they are applied to other types of PDEs, discretization schemes suitable for the underlying physical PDEs should be used.

An outline of the book is as follows.

Chapter 1 provides an introduction to moving mesh methods in one spatial dimension (1D). Specifically, simple moving mesh finite difference and finite element methods for time-dependent PDEs are described and implemented in Matlab. (The Matlab codes are available online on the first author's homepage.) The basic components of moving mesh methods – the mesh movement strategy, the PDE discretization for moving meshes, and the overall solution procedure – are discussed at the end of the chapter.

Chapter 2 provides a more detailed treatment of these three components of moving mesh methods in 1D. The well-known equidistribution principle plays a fundamental role in the design of mesh movement strategies throughout. The optimality properties of the equidistributing meshes (discrete case) and coordinate transformations (continuous case) are discussed. Using equidistribution, a number of mesh equations and moving mesh equations (MMPDEs) – continuous forms of mesh movement strategies formulated in terms of coordinate transformations – are developed for steady-state and time-dependent problems, respectively. Practical implementation issues, including discretization of mesh equations and physical PDEs and the overall solution procedure, are addressed. In particular, the discretization of the physical PDEs for a moving mesh using finite differences or finite elements can be done using either the quasi-Lagrange approach or the rezoning approach, and the coupled system of mesh and physical PDEs can be solved either simultaneously or alternately.

The key to the success of moving mesh methods lies in a suitable choice of a mesh density function. This function controls mesh concentration through the equidistribution principle and typically measures the difficulty in the spatial numerical approximation of the underlying problem. Several sections in this chapter are devoted to the selection of the mesh density function based on an interpolation error estimate, on scaling invariance, or on an a posteriori error estimate, with the optimal bound for interpolation error or solution error also obtained for the corresponding equidistributing mesh. Numerical results for a number of nontrivial physical examples of time-dependent PDEs are included.

The remaining chapters discuss and expand upon mesh adaptivity issues in the considerably more challenging multidimensional contexts. For these higher spatial dimensions, one needs the basic tools of advanced calculus to transform PDEs between the physical space and the computational space, and these are reviewed in Chapter 3. For the situation where the time-varying mesh is assumed to be known, a detailed discussion of implementation issues for the finite difference and finite element methods are then given for both the quasi-Lagrange and rezoning approaches.

Chapters 4 and 5 for the most part generalize the steady-state mesh adaptation strategies in Chapter 2 to multidimensions. The situation with mesh adaptivity now becomes much more complicated. The equidistribution principle, specifying only the volume of mesh elements, is no longer sufficient for determining a multidimensional mesh. An additional condition is needed for specifying the shape and orientation of mesh elements. Chapter 4 presents the basic principles of multidimensional mesh adaptivity, including this needed alignment condition. The mesh adaptivity is driven by a solution-dependent monitor function (a symmetric positive definite matrix function related naturally to the 1D mesh density function), which defines a metric on the physical domain. A fundamental interpretation of multidimensional mesh adaptivity is as a technique to generate an M-uniform mesh (a uniform mesh in

the metric space), which in turn provides a natural control of mesh equidistribution and alignment, the role of the latter being to ensure that the mesh is properly aligned with behavior of the physical solution. These equidistribution and alignment conditions are analyzed and related to the mesh quality in a mathematically precise way. Interpretations are given from both a discrete (mesh) and continuous (coordinate transformation) perspective.

Chapter 5 discusses how to choose an optimal monitor function for a given interpolation error bound or a posteriori error estimate, as well as a monitor function based upon other geometric and physical considerations. The rigorous theoretical treatment of interpolation error in general Sobolev spaces, given for both the isotropic and anisotropic cases, provides a strong result showing how the optimal choice of monitor function leads to an error bounded by an optimal solution-dependent factor times an optimal order of $1/N$, with N being the number of mesh elements. Interpolation error bounds associated with non-optimal monitor functions and optimal monitor functions for a posteriori error bounds are also addressed. Furthermore, various practical aspects of computing monitor functions are discussed.

The final two chapters are devoted to a discussion of the specific mesh adaptation strategies. For some, the development has been directly driven by the equidistribution and alignment conditions, while many others can be shown to be closely related. Even more generally, these two conditions can be used to facilitate an understanding of virtually all of them. The mesh adaptation strategies are separated into two loose categories, variational methods and velocity-based methods, which are addressed in Chapters 6 and 7, respectively. The variational methods determine the coordinate transformation needed for mesh generation as a minimizer of an adaptation functional typically designed to measure difficulty in the numerical simulation and to achieve desired mesh properties such as smoothness. Various theoretical and practical issues for the functionals are dealt with, including the existence and invertibility of minimizers, derivation of the corresponding Euler-Lagrange equations, and their finite difference and finite element discretization. A general MMPDE strategy is also developed from the steady-state adaptation functionals. A major effort in the chapter has been made to describe and analyze the large variety of variational methods which were originally developed based upon radically different motivations. Notably, a number of these methods, while designed in the context of mesh generation, can be very easily modified to perform mesh adaptation through proper insertion of a monitor function. Numerical examples are chosen to both illustrate strengths and weakness of the methods and to demonstrate their utility for solving nontrivial physical problems.

Chapter 7 discusses velocity-based methods, which target the mesh velocity directly and subsequently determine mesh point locations by integrating the velocity field. These are Lagrange type methods, being more or less motivated by Lagrangian methods in fluid dynamics for which the mesh coordinates are obtained as particle

trajectories by integrating a flow velocity. The first method considered, the GCL (Geometric Conservation Law) method, forces the mesh velocity field to satisfy an equidistribution condition and a curl condition. It shares common features with a method in Chapter 6 based upon the Monge-Kantorovich optimal mass transform problem, where the mapping itself is forced to satisfy such conditions. The GCL method is shown to be closely related to Lagrangian methods in fluid dynamics and a moving mesh method based on deformation maps. Two finite element methods, one based upon GCL and the other the original version of the moving finite element method (MFE), are discussed. Finally, some other physically motivated velocity-based methods are described.

Given the comprehensive treatment of the topics, navigating the book in a way which best suits a reader's interests can be a difficult task, and with this in mind we provide some guidance. While the analysis on optimal error bounds for adaptive meshes found in §§2.4, 2.9, 5.1, 5.2, 5.4, and 6.2 is fairly technical and intimidating for those lacking expertise in the mathematical theory of finite element methods, we want to emphasize the importance of the fundamental results in those sections. Nevertheless, some first time readers may wish to skip the above mentioned sections and focus on the summaries given at the end of §2.4 and §5.2 to avoid being caught up in unfamiliar technical details. In general, the reader may often skip sections not directly related to their research interest. For example, a reader mainly interested in theoretical aspects of adaptivity could skip most of the sections devoted exclusively to implementation issues such as §§2.6, 3.2, 3.4, 5.3, and 6.3. As well, readers mainly interested in finite element methods may skip those sections on finite difference methods and discretizations.

The book can be used as a textbook for an advanced, semester-long course in the numerical solution of partial differential equations. Such a course could cover the basic principles of adaptive mesh movement in 1D, higher dimensional discretization for PDEs on moving meshes, and general principles of mesh adaptation in Chapters 1–4, discussion of monitor functions in §5.2.5 and §5.3, and explanation of variational methods in §6.1 and §6.3. The methods described in Chapters 6 and 7 can be selected based upon the particular interests of the students and/or the instructor, although we would encourage using some treatment of the equidistribution and alignment principles in §6.4.

Finally, it is fair to say that moving mesh methods as a whole are still in a relatively early phase of development. Many of them are at the experimental stage, and almost all require further mathematical justification. Rigorous analysis of moving mesh methods for solving time-dependent PDEs has only been carried out for some very simple model problems to date, and more ways to improve their efficiency and robustness will no doubt be developed. For example, more systematic numerical

studies on how to reduce the costs in solving the overall system of mesh and physical PDEs are needed, not to mention how to balance time stepping with spatial mesh adaptation – a question that has received very little attention in the literature. It is the authors' hope that this book will serve as a springboard for the reader who wishes to learn and master basic methods for mesh adaptation in general, and mesh movement in particular, and that it will serve as a stepping stone toward a more complete understanding of and development of practicable moving mesh methods.

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Lawrence, Kansas and Vancouver, British Columbia *Weizhang Huang*
July, 2010 *Robert D. Russell*

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