AN INTRODUCTION TO MESHFREE METHODS AND THEIR PROGRAMMING

An Introduction to Meshfree Methods and Their Programming

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To Zuona

Yun, Kun, Run,

and my family

for the support and encouragement

G. R. Liu

To Qingxia

and Zhepu

for the love, support and encouragement

To my mentor, Professor Liu for his guidance

Y. T. Gu

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Preface

The finite difference method (FDM) has been used to solve differential equation systems for centuries. The FDM works well for problems of simple geometry and was widely used before the invention of the much more efficient, robust *finite element method (FEM)*. FEM is now widely used in handling problems with complex geometry. Currently, we are using and developing even more powerful numerical techniques aiming to obtain more accurate approximate solutions in a more convenient manner for even more complex systems. The meshfree or meshless method is one such phenomenal development in the past decade, and is the subject of this book.

There are many MFree methods proposed so far for different applications. Currently, three monographs on MFree methods have been published.

- *Mesh Free Methods, Moving Beyond the Finite Element Method* by GR Liu (2002) provides a systematic discussion on basic theories, fundamentals for MFree methods, especially on MFree weak-form methods. It provides a comprehensive record of well-known MFree methods and the wide coverage of applications of MFree methods to problems of solids mechanics (solids, beams, plates, shells, etc.) as well as fluid mechanics.
- *The Meshless Local Petrov-Galerkin (MLPG) Method* by Atluri and Shen (2002) provides detailed discussions of the meshfree local Petrov-Galerkin (MLPG) method and its variations. Formulations and applications of MLPG are well addressed in their book.
- *Smooth Particle Hydrodynamics; A Meshfree Particle Method* by GR Liu and Liu (2003) provides detailed discussions of MFree particle methods, specifically smoothed particle hydrodynamics (SPH) and some of its variations. Applications of the SPH method in fluid mechanics, penetration, and explosion have also been addressed in this book, and a general computer source code of SPH for fluid mechanics is provided.

Readers may naturally question the purpose of this book and the difference between this book and others, especially that by GR Liu (2002).

The second and the third books are related to specific MFree methods, which have clearly different scopes from this book. The book by GR Liu (2002) is the first book published with a comprehensive coverage on many major MFree methods. It covers all the relatively more mature meshfree methods based on weak-form formulations with systematic description and broad applications to solids, beams, plates, shell, fluids, etc. However, the starting point in that book is relatively high. It requires a relatively strong background on mechanics as well as numerical simulations. In addition, some expressions in this book were not given in detail, and no computer source code was provided, because of space limitation.

After the publication of the first book, the first author received many constructive comments, including requests for source codes and for more detailed descriptions on fundamental issues. This book is therefore intended to complement the first book and provide the reader with more details of the fundamentals of meshfree methods accompanied with detailed explanation on the implementation and coding issues together with the source codes. This book covers only the very basics of meshfree weak-form methods, but provides intensive details on meshfree methods based on the strong-form and weak-strong-form formulations. The relationship of this book and the book by GR Liu (2002) is detailed in Table 0.1. This shows that there is very little duplication of materials between the two; they are complementary. The authors hope that this monograph will help beginning researchers, engineers and students have a smooth start in their study and further exploration of meshfree techniques.

The purpose of this book is, hence, to provide the fundamentals of MFree methods in as much detail as possible. Some typical MFree methods, such as EFG, MLPG, RPIM, and LRPIM, are discussed in great detail. The detailed numerical implementations and programming for these methods are also provided. In addition, the MFree collocation (strong-form) methods are also detailed. Many well-tested computer source codes for MFree methods are provided. The application and the performance of the codes provided can be checked using the examples attached. Input and output files are provided in table form for easy verification of the codes. All computer codes are developed by the authors based on existing numerical techniques for FEM and the standard numerical analysis. These codes consist of most of the basic MFree techniques, and can be easily extended to other variations of more complex procedures of MFree methods.

Releasing this set of source codes is to suit the needs of readers for an easy comprehension, understanding, quick implementation, practical applications of the existing MFree methods, and further improvement and

Table 0.1. The relationship between this book and the meshfree method book by GR Liu (2002)

Topics	**Book by GR Liu (2002)**		**This book**	
	Content	Source code	Content	Source code
Weighted residual methods	Briefed	NA	Detailed explicitly with 1D examples	NA
Weak-forms	Detailed	NA	Briefed	NA
MFree shape functions	Detailed with emphasizes on MLS, PIM and RPIM	No	Detailed for MLS, PIM WLS, RPIM, and Hermite-type	Provided
MFree global weak-form methods	Detailed for EFG, PIM and RPIM	No	Detailed for EFG and RPIM	Provided
MFree local Petrov-Galerkin weak-form methods	Detailed for MLPG, LPIM and LRPIM	No	Detailed for MLPG and LRPIM	Provided
MFree collocation methods	No	No	Detailed for various techniques	No
MFree weak-strong form methods	No	No	Detailed for MWS-LS and MWS-RPIM	Provided
Boundary-type MFree methods	Detailed for BPIM and BRPIM	No	No	NA
Coupled methods	Detailed for EFG/BEM, MLPG/FEM/BEM	No	No	NA
SPH	Detailed for fluid mechanics problems	No	No	NA
Applications to solids	1D and 2D solids	No	1D, 2D and 3D solids	Partially provided
Applications to beam, plate and shell structures	Yes	No	No	NA
Applications to fluid mechanics problems	Detailed for SPH, MLPG and LRPIM	No	Detailed using MWS	No
Material non-linear problems	Yes	No	No	NA
Geometric non-linear problems	No	NA	Provided examples of RPIM	No
Convection-dominated problems	No	No	Detailed for 1D and 2D problems using MFree strong-form methods	No
MFree2D©	Detailed for usage and techniques used	No	No	NA

NA: not applicable.

development of their own MFree methods. All source codes provided in this book are developed and tested based on the MS Windows and MS Developer Studio 97 (Visual FORTRAN Professional Edition 5.0.A) on a personal computer. After slight revisions, these programs can also be executed in other platforms and systems, such as the UNIX system on workstations. In our research group these codes are frequently ported between the Windows and UNIX systems, and there has been no technical problem.

Outline of this book

Chapter 1: The weighted residual methods are introduced and discussed. Various numerical approaches derived from the weighted residual method are introduced and examined using 1D examples. The fundamental and theories of solid mechanics and weak-forms are also provided.

Chapter 2: An overview of MFree methods is provided, including the background, classifications, and basic procedures in MFree methods.

Chapter 3: Fundamental and theories of MFree interpolation /approximation schemes for shape function construction, especially, MLS, PIM, WLS, and RPIM, and Hermite-type shape functions, are systemically introduced. Source codes of two standard subroutines of computing MLS and RPIM shape functions are provided.

Chapter 4: Formulations of the MFree global weak-form methods, EFG and RPIM, are presented in detail. A standard source code of RPIM and EFG is provided.

Chapter 5: Formulations of the MFree local weak-form methods, MLPG and LRPIM, are presented in great detail. A standard source code of LRPIM is provided.

Chapter 6: Fundamentals and procedures of the MFree collocation methods are systemically discussed. The issues related to the stability and accuracy in the strong-form methods are discussed in detail. In particular, the effects of the presence of the derivative boundary conditions are examined in great detail.

Chapter 7: The MFree methods based on combination of local weak form and collocation are derived and discussed in detail. A standard source code is provided.

The book is written for senior university students, graduate students, researchers, professionals in engineering and science. Readers of this book can be any one from a beginner student to a professional researcher as well as engineers who are interested in learning and applying MFree methods to solve their problems. Knowledge of the finite element method is not required but it would help in the understanding and comprehension of many concepts and procedures of MFree methods. Basic knowledge of solids mechanics would also be helpful. The codes provided for practise might be the most effective way to learn the basics of MFree methods.

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